Faculty of Science, Technology, Engineering and Mathematics
MST210 Mathematical methods, models and modelling

MST210

TMA 08 2019J

Covers Units 19, 20, 21 and revision **Cut-off date 13 May 2020**

The module website gives details of how to submit assignments for this module. If you have a disability that makes it difficult for you to attempt any of the questions, then please contact your Student Support Team or your tutor for advice.

The marks allocated to each part of each question are indicated in brackets in the margin.

You may use Maxima to check answers, but it should not be used otherwise to answer TMA questions except where explicitly indicated. TMA answers must show full working to attract full marks.

In order to encourage you to present your solutions to the TMA questions in a good mathematical style, your tutor will comment on how you:

- use correct mathematical notation
- define any symbols that you introduce in formulating and solving a problem
- give references for standard formulas and derivations
- include comments and explanations within your mathematics
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question
- draw diagrams and graphs.

These features are seen as being essential to complementing your mathematical skills. Your tutor will make comments on how well you achieve these objectives and give you guidance on how to satisfy them.

Five of the marks for this TMA will be allocated to the way you write your solutions. It is expected that most students will receive the majority of these presentation marks; such marks are included in TMAs to encourage, and emphasise the need for, thinking about how you present your mathematics.

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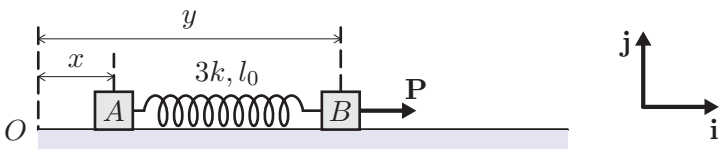
Question 1 (Unit 19) – 16 marks

Two particles A and B of masses m_1 and m_2, respectively, are attached to a model spring of natural length l_0 and stiffness $3k$. The particles rest on a rough horizontal table; the coefficient of sliding friction between the particles and the table is μ. At time $t = 0$ the system is at rest with the spring at its natural length, and particle A is at the point O. Choose unit vectors with **i** horizontal and **j** vertical as shown in the diagram below. A constant horizontal force **P** of magnitude P directed along **i** is applied to particle B, and at subsequent times t the positions of the particles (measured from the fixed point O) are x and y, as shown in the diagram.



(a) Write down the position of the centre of mass of the two particles in terms of x and y. [2]

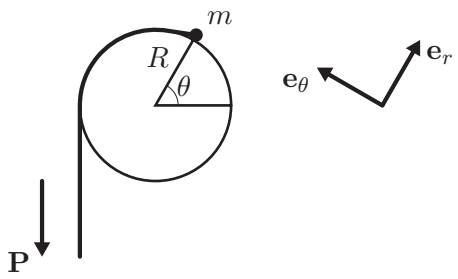
(b) Draw a force diagram for each particle. Express all forces in vector form, giving magnitude and direction, and state clearly which forces are internal to the system and which are external. [5]

(c) Determine the equation of motion of each particle. [6]

(d) Use your results to find the equation of motion of the centre of mass of the two particles. Verify that this equation is equivalent to treating the system as a particle of mass $m_1 + m_2$ that is subject only to the external forces. [3]

Question 2 (Unit 20) – 17 marks

The axis of a smooth fixed circular cylinder of radius R is horizontal. A particle of mass m is attached to a model string and is initially at rest level with the centre of the cylinder with the string draped over the top, where it slides without friction as if on a model pulley, as shown in the diagram below. A constant force **P** of magnitude P pulls the model string downwards. Let θ denote the angle subtended at the axis of the cylinder between the initial position of the particle and its current position. Let $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be the radial and tangential unit vectors, as shown in the diagram.



(a) Draw a force diagram showing all the forces acting on the particle. [2]

(b) Express each force in terms of the unit vectors shown in the diagram. [3]

(c) Derive the equation of motion of the particle. Resolve radially and tangentially to show that the tangential component is given by

$$\ddot{\theta} = \frac{P - mg\cos\theta}{mR},$$

where g is the magnitude of the acceleration due to gravity. [5]

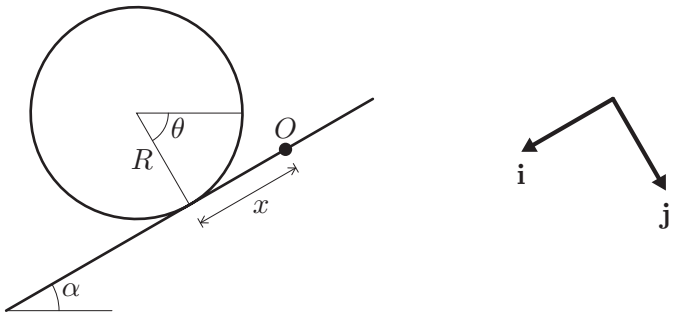
(d) By multiplying the equation in part (c) by $\dot{\theta}$ and integrating with respect to t, determine $\dot{\theta}^2$ as a function of θ, and hence show that the magnitude of the normal reaction of the cylinder on the particle is

$$3mg\sin\theta - 2P\theta.$$ [4]

(e) Using Maxima, or otherwise, determine the angle to the horizontal, to the nearest degree, at which the particle leaves the surface of the cylinder in the case where $P = \frac{7}{6}mg$. [3]

Question 3 (Unit 21) – 17 marks

A thin spherical shell of radius R and mass M rolls down a slope that is inclined at an angle α to the horizontal. The sphere starts from rest and rolls without slipping a distance L down the slope. Choose x to be measured down the slope, starting from a point O where the sphere starts from rest. Measure θ anticlockwise from the normal to the slope as shown in the diagram, so that $x = R\theta$ as the sphere rolls down the slope. Choose unit vectors **i** pointing along Ox down the slope, and **j** at an angle $\pi/2$ anticlockwise (the same sense as increasing θ), as shown in the diagram.



(a) Draw a force diagram showing all of the forces acting on the sphere, and mark the point of action of each force. [3]

(b) Apply Newton's second law to the sphere to obtain an equation for the linear acceleration $\ddot{x}$ of the sphere in terms of the magnitude F of the friction force at the point of contact, and other parameters. [2]

(c) Calculate the torque of each force about the axis of rotation of the sphere. [3]

(d) Apply the equation of relative rotational motion to the sphere to show that the acceleration of the sphere is $\frac{3}{5}g\sin\alpha$, where g is the magnitude of the acceleration due to gravity. [6]

(e) Calculate the time t for the sphere to roll a distance L down the slope without slipping, in terms of known constants. [3]

The following revision questions are in the style of Section 2 of the examination.

Question 4 (Revision: Unit 1) – 5 marks

Consider the differential equation

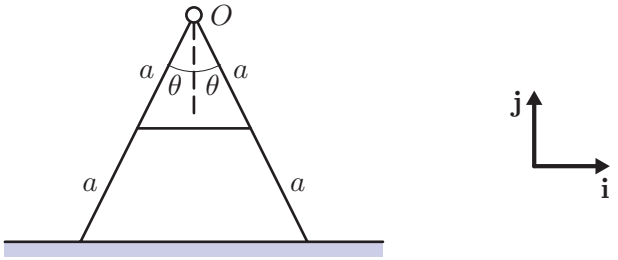
$$\frac{dy}{dx} = \frac{e^{-y}}{x}, \quad x \geq 1.$$

(a) Find the general solution of the differential equation explicitly in the form $y = f(x)$. [3]

(b) Find the particular solution that satisfies $y(1) = 0$. [2]

Question 5 (Revision: Unit 2) – 5 marks

A pair of stepladders is modelled as two uniform model rods, each of length $2a$ and mass m, that are freely hinged together at the top end (point O). The bottom ends of the rods rest on a smooth horizontal floor. Both rods are inclined at an angle θ to the downward vertical, with their midpoints joined by a model string. Choose horizontal and vertical unit vectors **i** and **j**, as shown in the diagram below.



(a) Draw a force diagram showing the external forces acting on the stepladders, and hence determine the total normal reaction force acting on them due to the floor. [2]

(b) Now consider the model rod that models the left-hand ladder. By symmetry, you may assume that half of the total normal reaction force acts on this ladder. Take torques about O and hence determine the magnitude of the tension in the model string in terms of the above parameters and g, the magnitude of the acceleration due to gravity. [3]

Question 6 (Revision: Unit 5) – 5 marks

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 6 & -8 \\ -2 & 5 & -4 \\ -1 & 1 & 1 \end{pmatrix}.$$

(a) Given that the matrix **A** has an eigenvector $\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$, calculate the corresponding eigenvalue. [2]

(b) Given that one of the eigenvalues of the matrix **A** is 2, calculate a corresponding eigenvector. [3]

Question 7 (Revision: Unit 7) – 5 marks

Consider the function

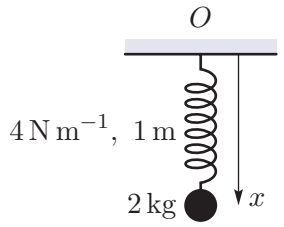
$$f(x, y) = 2x^2 - xy^2 + 6xy - 5x + 2.$$

(a) Find the stationary points of f. [3]

(b) Classify any stationary points (x, y) with $x \neq 0$ and $y \neq 0$. [2]

Question 8 (Revision: Unit 9) – 5 marks

A particle of mass 2 kg is attached to one end of a model spring that is hanging vertically from a fixed point O. The spring has stiffness $4\,\mathrm{N\,m^{-1}}$ and natural length 1 m. The system is oscillating in a vertical line with the particle below O. In this question use the approximation that the magnitude of the acceleration due to gravity is $10\,\mathrm{m\,s^{-2}}$. Take the point O as the datum for potential energy, and measure x downwards as shown in the diagram below.

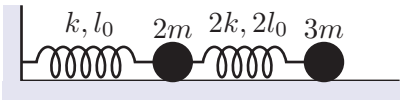


(a) Find an expression for the total mechanical energy function for the system. [2]

(b) When the particle is 2 m below O, it has speed $2\,\mathrm{m\,s^{-1}}$. Use conservation of mechanical energy to establish whether the spring is ever in compression during the motion. (*Hint*: Try to determine the speed of the particle when the spring has its natural length.) [3]

Question 9 (Revision: Unit 11) – 5 marks

A particle of mass $2m$ is connected on the left to a fixed wall by a spring of stiffness k and natural length l_0. This particle is connected to a second particle of mass $3m$ by a spring of stiffness $2k$ and natural length $2l_0$, as shown in the diagram below. The particles are constrained to move in a straight line on a smooth horizontal surface, and their displacements from their equilibrium positions are x and y, respectively.



(a) Draw a force diagram showing all the forces acting on each particle. [1]

(b) Derive the equation of motion for each particle. [4]

Question 10 (Revision: Unit 13) – 5 marks

Consider the periodic function defined by

$$f(t) = \begin{cases} t^2 & \text{for } -1 < t \leq 0, \\ t & \text{for } 0 < t \leq 1, \end{cases}$$
$$f(t+2) = f(t),$$

and its Fourier series

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t) + \sum_{n=1}^{\infty} B_n \sin(n\pi t).$$

(a) Sketch the function $f(t)$ over the range $-3 \leq t \leq 3$, and hence state whether it is even, odd, or neither even nor odd. [3]

(b) Calculate the constant term A_0. [2]

Question 11 (Revision: Unit 17) – 5 marks

A hemisphere D of radius R is situated with centre at the origin and lying above the xy-plane. Consider the function of three variables given in spherical coordinates as $f(r, \theta, \phi) = r + \cos\phi$.

Calculate the volume integral I given by

$$I = \int_D f(r, \theta, \phi)\, dV.$$ [5]

Question 12 (Revision: Unit 19) – 5 marks

Two particles of masses $2m$ and m lie in a smooth horizontal groove, in which they are constrained to move. Let $\mathbf{i}$ be a unit vector directed along the groove. The particle of mass $2m$ is given a velocity $6\mathbf{i}$ so that it undergoes an elastic collision with the other particle, which is initially at rest. Calculate the velocity of the two particles after the collision. [5]